Received: *July 3, 1998*, Accepted: *August 27, 1998*

Remarks on the heavy quark potential in the supergravity approach

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Abstract: We point out certain unexpected features of the planar QCD_3 confining potential, as computed from a classical worldsheet action in an AdS metric via the Maldacena conjecture. We show that there is no Lüscher c/R term in the static-quark potential, which is contrary to both the prediction of various effective string models, and the results of some recent lattice Monte Carlo studies. It is also noted that the glueball masses extracted from classical supergravity tend to finite, coupling-independent constants in the strong coupling limit, even as the string tension tends to infinity in the same limit; this is a counter-intuitive result.

Keywords: Strong Coupling Expansion, Confinement, Black Holes in String Theory, Brane Dynamics in Gauge Theories.

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JHEP08(1998)009

The startling possibility that continuum QCD may be solved (albeit in the large-N and strong-coupling limits) via classical supergravity has motivated a great deal of effort in recent months. Following the seminal proposal of Maldacena [1, 2] and further developments by Witten [3, 4], there have been explicit calculations of the string tension [5, 6] and glueball mass spectrum [7, 8] of non-supersymmetric planar QCD in D=3 and D=4 dimensions from the supergravity approach. A number of qualitative features of planar QCD have been derived, such as the confinement/deconfinement transition at finite temperature [4], screening of magnetic charge and representation-dependence of Wilson loops [9], and the length-law falloff of 't Hooft loops [10], and these agree with our expectations. The glueball mass ratios, although computed at strong couplings, have even been favorably compared with corresponding lattice Monte Carlo results, extrapolated to the continuum and large-N by Teper [12]. The supergravity solution of planar QCD thus appears to be an explicit realization of the large-N master field.

In this article we will be concerned with certain qualitative features of the heavy quark potential in the supergravity approach. It should be noted that in computing the heavy quark potential there are two types of contributions: those that come from the non-trivial background metric (the "classical" worldsheet), and those that come from quantum fluctuations of the worldsheet. In the present paper shall concentrate on the former, although at the end of the paper we will also comment on the latter.

Our main point is that there are two features of the heavy quark potential in planar QCD_3, extracted along the lines of refs. [4, 5, 6], which do not agree with the expected behavior of the continuum theory. These are the absence of a Lüscher term [11] in the potential, and the fact that the glueball mass spectrum is almost independent of the string tension. We hasten to add that these points do not necessarily undermine the validity of the Maldacena conjecture. It only means that there is some aspect of either the large-N or strong-coupling limits, as taken in the supergravity approach, which causes the solution to differ qualitatively from finite-N, asymptotically free gauge theory.

We begin by calculating, in the supergravity approach, the subleading term in the static quark potential at large quark separation As explained in refs. [2, 4, 5, 6], the heavy quark potential in planar QCD_3 at strong-coupling is obtained by calculating the action of a spacelike classical Nambu-Goto string in a Wick-rotated $AdS_5 \times S_5$ black-hole metric [1]

$$ds^2 = \alpha' \left\{ \frac{U^2}{R^2}[f(U)dt^2 + dx_i^2] + \frac{R^2}{U^2} f^{-1}(U) dU^2 + R^2 d\Omega_5^2 \right\} , \tag{1}$$

where the boundary C of the string worldsheet lies at $U = \infty$, and

$$f(U) = 1 - \frac{U_T^4}{U^4}, \qquad R^2 = \sqrt{4\pi g_s N} = \sqrt{4\pi g_{YM}^2 N} \, . \tag{2}$$

[1]We use throughout the notation of ref. [5].

It will also be useful below to make the rescaling

$$U = \sqrt{4\pi g_s N}\rho \ , \quad U_T = \sqrt{4\pi g_s N} b \tag{3}$$

and express the metric in the form

$$ds^2 = \alpha' \sqrt{4\pi g_s N} \left\{ \frac{d\rho^2}{\rho^2 - \frac{b^4}{\rho^2}} + (\rho^2 - \frac{b^4}{\rho^2}) dt^2 + \rho^2 dx_i^2 + d\Omega_5^2 \right\} . \tag{4}$$

At length scales much less that b^{-1}, the supergravity solution is probing planar $\mathcal{N} = 4$ super Yang-Mills theory in D=4 dimensions, while at scales above b^{-1}, the solution should probe planar non-supersymmetric Yang-Mills theory in D=3 dimensions. The mass scale b can therefore be thought of as an ultraviolet regulator for the D=3 dimensional theory. The proposal of refs. [2, 4, 5, 6] is that the expectation value of the Wilson loop at large N is given by $W(C) \sim \exp(-S)$, where the exponent is the action of the classical worldsheet bounding loop C. This leads to the following implicit expressions for the static quark potential E and a function of quark separation L

$$L = 2\frac{R^2}{U_0} \int_1^\infty \frac{dy}{\sqrt{(y^4-1)(y^4-1+\epsilon)}}, \tag{5}$$

and

$$\begin{aligned} E &= \frac{U_0}{\pi} \int_1^\infty dy \left(\frac{y^4}{\sqrt{(y^4-1)(y^4-1+\epsilon)}} - 1 \right) + \frac{U_0 - U_T}{\pi} \\ &= \frac{U_0^2}{2\pi R^2} L + \frac{U_0}{\pi} \int_1^\infty dy \left(\sqrt{\frac{y^4-1}{y^4-1+\epsilon}} - 1 \right) + \frac{U_0 - U_T}{\pi}. \end{aligned} \tag{6}$$

Here

$$\epsilon = f(U_0) \ll 1, \text{ so } U_0 \approx U_T = \pi R^2 T = \sqrt{4\pi^3 g_s N} T. \tag{7}$$

We now want to obtain the next to leading behavior of E. To this end, consider the integral

$$J(\epsilon) \equiv \int_1^\infty dy \left(\sqrt{\frac{y^4-1}{y^4-1+\epsilon}} - 1 \right). \tag{8}$$

Clearly $J(0) = 0$. In order to obtain an asymptotic expansion for small ϵ, it turns out to be most convenient first to differentiate $J(\epsilon)$,

$$\frac{\partial J(\epsilon)}{\partial \epsilon} = -\frac{1}{2} \int_1^\infty \frac{\sqrt{y-1}}{(y-1+\epsilon/4)^{3/2}} \Phi(y), \tag{9}$$

where $\Phi(y)$ is given by

$$\Phi(y) \approx \frac{\sqrt{(y+1)(y-i)(y+i)}}{(y+1-\epsilon/4)^{3/2}(y-i+i\epsilon/4)^{3/2}(y+i-i\epsilon/4)^{3/2}}, \tag{10}$$

which is regular for $y = +1$ and/or $\epsilon = 0$. To obtain the asymptotic expansion, we perform a partial integration,

$$\frac{\partial J(\epsilon)}{\partial \epsilon} = -\frac{1}{2}\left[\left\{-2\sqrt{\frac{y-1}{y-1+\epsilon/4}} + 2\ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4})\right\}\Phi(y)\right]_1^\infty + \\ + \int_1^\infty dy\left(-\sqrt{\frac{y-1}{y-1+\epsilon/4}} + \ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4})\right)\Phi'(y). \qquad (11)$$

Thus we have

$$\frac{\partial J(\epsilon)}{\partial \epsilon} = \frac{1}{8}\ln\epsilon + \int_1^\infty dy\left(-\sqrt{\frac{y-1}{y-1+\epsilon/4}} + \ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4})\right)\Phi'(y) + \\ +O(\epsilon). \qquad (12)$$

Making a further partial integration in the integral on the right hand side of this equation, using

$$\int dy\left(-\sqrt{\frac{y-1}{y-1+\epsilon/4}} + \ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4})\right) = \\ = -\frac{3}{2}\sqrt{(y-1)(y-1+\epsilon/4)} + (y-1+\frac{3\epsilon}{8})\ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4}), \qquad (13)$$

we see that

$$\int_1^\infty dy\left(-\sqrt{\frac{y-1}{y-1+\epsilon/4}} + \ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4})\right)\Phi'(y) = O(\epsilon\ln\epsilon). \qquad (14)$$

Collecting our results, we thus obtain

$$\frac{\partial J(\epsilon)}{\partial \epsilon} = \frac{1}{8}\ln\epsilon + O(\epsilon\ln\epsilon). \qquad (15)$$

Since $J(0) = 0$, we can integrate to obtain

$$J(\epsilon) = \frac{1}{8}\epsilon\ln\epsilon + O(\epsilon^2\ln\epsilon). \qquad (16)$$

In order to get a physical interpretation of this, we need to express L in terms of ϵ. From eq. (5) we have

$$L \approx 2\frac{R^2}{U_T}\int_1^\infty \frac{dy}{\sqrt{(y-1)(y-1+\epsilon/4)}}\frac{1}{\sqrt{F(y)}}, \qquad (17)$$

where the function $F(y)$ is given by

$$F(y) = (y+1)(y-i)(y+i)(y+1-\epsilon/4)(y-i+i\epsilon/4)(y+i-i\epsilon/4). \qquad (18)$$

This function does not vanish for $y = 1$ and/or $\epsilon = 0$. Hence it does not produce any singularity in the integral in eq. (17). To obtain an asymptotic expansion of L, we proceed as before by a partial integration in eq. (17), using

$$\int \frac{dy}{\sqrt{(y-1)(y-1+\epsilon/4)}} = 2\ln(\sqrt{y-1}+\sqrt{y-1+\epsilon/4}). \qquad (19)$$

We thus obtain to the leading order

$$L \approx -\frac{R^2}{2U_T} \ln \epsilon + O(\epsilon \ln \epsilon). \tag{20}$$

Hence the energy becomes

$$\begin{aligned} E &\approx \frac{U_T^2}{2\pi R^2} L \left(1 - \frac{1}{2} e^{-2U_T L/R^2}\right) \\ &\approx \frac{\sqrt{4\pi g_s N}}{2\pi} b^2 L \Big(1 - \frac{1}{2} e^{-2bL}\Big). \end{aligned} \tag{21}$$

The leading correction to the linear potential is thus exponentially small, for $Lb > 1$, of order $L \exp(-2Lb)$.

For QCD_4 a similar calculation can be performed, again using the results of [5, 6]. We shall not repeat the details, which are quite similar to those reported above, but we just give the final result,

$$E \approx \sigma L \left(1 - \frac{1}{2} e^{-bL}\right). \tag{22}$$

Again we see that the correction is exponentially small.

It should be emphasized that this result is valid also if the leading correction in $(4\pi g_{YM}^2 N)^{-1/2}$ is included. The first non-leading correction to the AdS_5 black hole metric was found in [13] to be

$$\frac{ds^2}{\alpha' \sqrt{4\pi g_s N}} = (1+\delta_1) \frac{d\rho^2}{\rho^2 - \frac{b^4}{\rho^2}} + (1+\delta_2)(\rho^2 - \frac{b^4}{\rho^2}) dt^2 + \rho^2 dx_i^2 + d\Omega_5^2, \tag{23}$$

where

$$\begin{aligned} \delta_1 &= -\frac{15}{8} \zeta(3) \alpha'^3 \left(5\frac{b^4}{\rho^4} + 5\frac{b^8}{\rho^8} - 3\frac{b^{12}}{\rho^{12}}\right), \\ \delta_2 &= +\frac{15}{8} \zeta(3) \alpha'^3 \left(5\frac{b^4}{\rho^4} + 5\frac{b^8}{\rho^8} - 19\frac{b^{12}}{\rho^{12}}\right). \end{aligned} \tag{24}$$

This will modify the integrals in L and E by polynomials in $1/y$, and they do not modify the crucial logarithmic singularity at $y = 1$. In particular, the leading correction to the linear potential is again of the type $\sigma L \exp(-\text{const.} \times L)$. Since the higher order corrections are expected to be of the polynomial type, there is not much chance of modifying the exponential approach to the linear potential.

In lattice QCD there have been many calculations of the heavy quark potential for various gauge groups, and it is safe to say that the linear asymptotic behavior is well established (see, e.g., the results of Bali et al. [14] for the case of D=4 and SU(2)). Current numerical evidence for the sub-leading Lüscher term $-c/L$ at large L in the interaction potential

$$E = \sigma L - c/L + \cdots \tag{25}$$

is quite convincing for the case of Z_2 lattice gauge theory in D=3 dimensions [15], but there are also strong indications of the existence of this term in the most recent (and, for us, more relevant) data for D=3 lattice SU(2) gauge theory [12]. The proposal (25) for the potential is inspired by string theories, where c is proportional to the central charge. For superstrings, $c = 0$, and hence the Lüscher term is absent for such strings. For bosonic strings we expect $c = (d-2)\pi/24$, if $E(L)$ is the quark potential extracted from Wilson loops, or $c = (d-2)\pi/6$ if $E(L)$ represents the mass of a flux tube created by a Wilson loop winding once through a periodic lattice of length L in the flux-tube direction [16]. The general conclusion is that for confining strings in lattice gauge theory, the constant c agrees fairly well with the values obtained in [11] and [16] for the bosonic string.

Comparing (25) to our result (21), we see that if the fits of the lattice Monte Carlo data are taken seriously, then there is a large discrepancy between the actual QCD string as seen on a lattice at weak couplings, and the QCD string obtained from supergravity. This could indicate the existence of a phase transition obtained in the supergravity approach as the effective Yang-Mills coupling $g_{YM}^2 N$ is reduced. This possibility has already been noted by Gross and Ooguri [9], and it is significant that such transitions are known to occur in lattice gauge theory. The strong and weak coupling regimes on the lattice are separated by a roughening phase transition; *in the strong coupling phase there is no Lüscher term, whereas this term does exist in the weak coupling phase.*

It should again be emphasized that the order by order corrections in $(4\pi g_{YM}^2 N)^{-1/2}$ do not improve the situation with respect to the discrepancy between the next to leading order potential. This is because the logarithmic singularity at $y = 1$ is not influenced by these polynomial corrections. Of course, it is a possibility that if the corrections could be computed non-perturbatively, then the situation might improve.

It goes without saying that there are always problems extracting sub-leading behavior from a fit like (25), since it is possible that a different type of fit may also reproduce the data quite well. It is therefore interesting to ask what result would be obtained if the same method for extracting c on the lattice were applied to potential derived from eqs. (5) and (6) above. Put another way, could the lattice results for c be obtained from a potential of the supergravity form? The method used in ref. [12] was to compute the quantity

$$c_{eff} = \frac{\frac{\mathcal{E}(\gamma L)}{\gamma L} - \frac{\mathcal{E}(L)}{L}}{\frac{1}{L^2} - \frac{1}{(\gamma L)^2}} \tag{26}$$

with $\gamma = 1.5$, where in this case $\mathcal{E}(L)$ is the energy of a flux tube state created by a spacelike Polyakov line, winding once through the periodic lattice. The signal for the Lüscher term is $c_{eff} \to c$ as $L \to \infty$, with c finite. In the numerical data presented in Table 9 of ref. [12], there is good evidence of a systematic rise in c_{eff} to a value consistent with $c = \pi/6$, which is the "universal" (i.e. bosonic string) value appropriate to mass of flux loops on the periodic lattice.

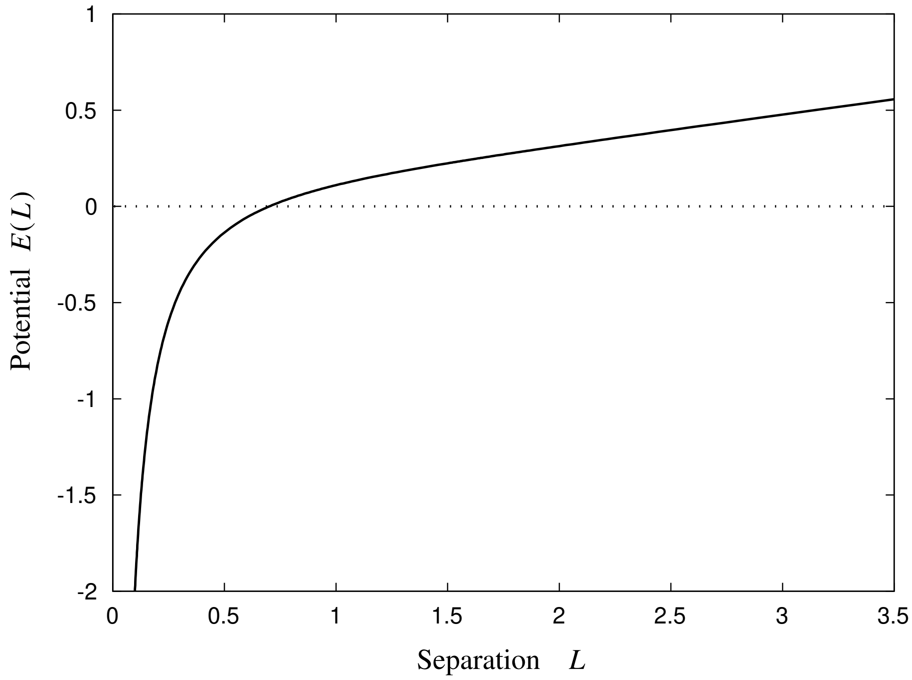

Figure 1: Heavy-quark potential determined from supergravity. At short distances $L << 1$, the potential is that of $\mathcal{N} = 4$ super Yang-Mills theory in D=4 dimensions, while for $L > 1$ it should match that of planar QCD_3 at strong coupling.

In fig. 1 we show a numerical solution of eqs. (5) and (6) for the static quark potential obtained from supergravity, where the axes display the rescaled, dimensionless values

$$\begin{aligned} E_{rs} &= \frac{E(L)}{\sqrt{4\pi g_s N} b}\,, \\ L_{rs} &= Lb\,. \end{aligned} \tag{27}$$

In the region $L_{rs} \ll 1$ we are probing $\mathcal{N} = 4$ Yang-Mills in D=4 dimensions, and the potential is Coulombic. Hence c_{eff} should be constant in this region. The region of interest is $L_{rs} > 1$, where the solution probes planar QCD at strong-coupling in D=3 dimensions. Figure 2 is a plot of c_{eff} vs. L_{rs}, extracted from the potential shown in Fig. 1. The point of this plot is that c_{eff} *drops* with increasing L; precisely the opposite behavior from what is reported in weak-coupling lattice gauge theory in D=3 dimensions [12]. Although this is hardly a conclusive argument (it is always possible that at yet larger distances the Monte Carlo data for c_{eff} will also start to drop), the existing evidence for flux-tube energy does seem to favor $c \neq 0$ over a potential of the form derived from supergravity.

Finally, we comment briefly on the glueball mass spectrum of D=3 planar Yang-Mills, derived in the supergravity approach in refs. [7, 8]. It is found that in the strong-coupling limit, glueball masses have the form

$$M_G = K\Big(1 + O[(g_{YM}^2 N)^{-3/2}]\Big) b\,, \tag{28}$$

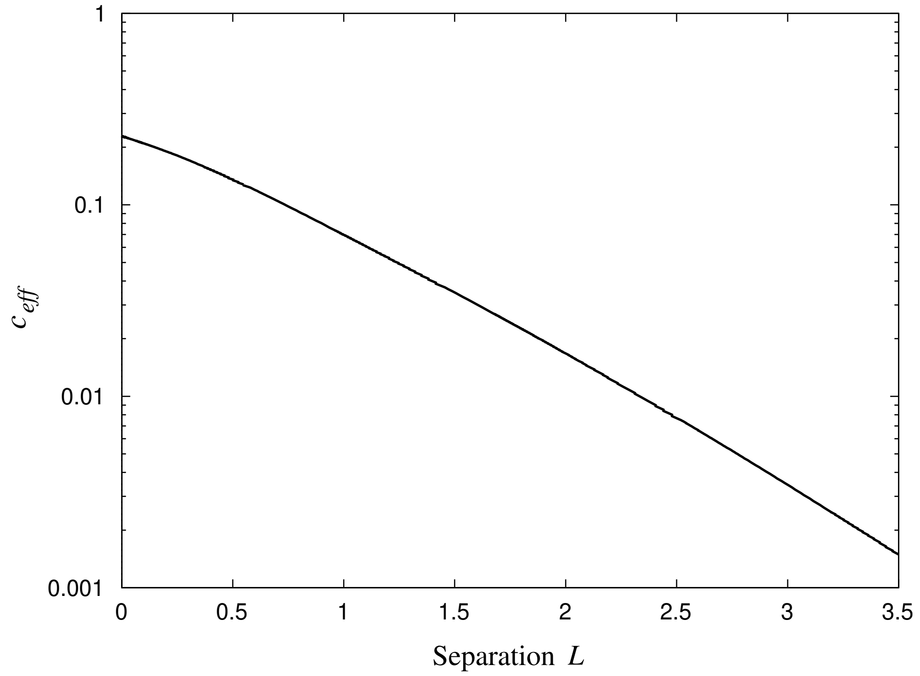

Figure 2: Searching for a Lüscher term where there is no Lüscher term: Calculation of c_{eff} in the heavy-quark potential obtained from supergravity.

where K is a pure number which depends on the quantum numbers of the glueball in question, but which is *independent of the gauge coupling.* Since the string tension obtained from (21) is

$$\sigma = \frac{\sqrt{4\pi g_s N}}{2\pi} b^2 \,, \tag{29}$$

the ratio

$$\frac{M_G}{\sqrt{\sigma}} \to 0 \tag{30}$$

tends to zero in the $g^2_{YM}N \to \infty$ limit. This is a rather counter-intuitive result. If glueballs are thought of as tubes of electric flux, with the string tension essentially the energy-per-unit-length of such flux tubes, then it is very natural to expect the glueball mass to increase as the string tension increases. This is certainly what happens in strong-coupling lattice theory. But apparently it is not what happens in the supergravity approach. We stress that this is a qualitative, rather than a numerical, issue; at strong-coupling there just seems to be no obvious relationship between the glueball mass spectrum and the strength of the confining force.

We conclude that both of the features noted here, namely, the absence of a Lüscher term in the heavy-quark potential, and the finiteness of the glueball mass in the limit of infinite string tension, suggest that the supergravity solution to D=3 planar Yang-Mills theory is probably not a very realistic representation of continuum QCD_3.[2] Of

[2]Certain other reservations about the glueball spectrum in the supergravity approach have been expressed in a very recent article by Ooguri et al. [17].

course, this solution was obtained in the strong-coupling limit. It is therefore reasonable to expect that as the gauge-coupling is reduced, a phase transition of some kind is encountered, as also suggested in ref. [9]. Below this transition, which is presumably associated with roughening, a finite Lüscher term can appear in the potential, and a more realistic relation between string tension and glueball masses may be obtained.

In this article we have addressed only the heavy quark potential extracted from the classical action, along the lines of refs. [2, 4, 5, 6]. In principle there could be the possibility that a Lüscher term might arise in going beyond the classical action, including also the quantum fluctuations of the worldsheet. The worldsheet, however, is that of a critical superstring. Then $c = 0$, at least naively, and one would not expect to get a Lüscher term from this source. However, there may still be a possibility, suggested to us by Ooguri [18], that worldsheet fluctuations in the neighborhood of the horizon could produce an effective $c > 0$. Whether a Lüscher term of the appropriate magnitude could be produced in this way remains to be seen.

Acknowledgments

We would like to thank Hirosi Ooguri for comments on an earlier version of this article. J.G.'s research is supported in part by Carlsbergfondet, and in part by the U.S. Department of Energy under Grant No. DE-FG03-92ER40711.

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